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CHAPTER

DIRAC DELTA FUNCTION

LEARNING OBJECTIVES

- Definition of Dirac Delta Function.
- Representation of Dirac Delta Function as limit of Gaussian Function and Rectangular Function.
- Other Representation of Dirac Delta Function. Properties of Dirac Delta Function.

2.1 DEFINITION OF DIRAC DELTA FUNCTION

In Mechanics, we often come across with situations in which a large force acts for a very short time interval. Thus concept of impulse is frequently encountered in many situations. So we can represent such cases with an impulse function called Dirac-Delta Function $\delta(x)$.

It is defined through the equation

$$\begin{aligned}\delta(x) &= 0 \quad \text{if } x \neq 0 \\ &= \infty \quad \text{if } x = 0\end{aligned} \quad \dots(1)$$

$$\text{and } \int_{-\infty}^{\infty} \delta(x) dx = 1 \quad \dots(2)$$

The region of integration includes the point $x = 0$.

The Dirac-Delta function is qualitatively meant as being zero everywhere except at $x = 0$ where it is infinitely high and the area between it and the x axis is finite and equal to unity. It is the limit of a sequence of function.

Also if $f(x)$ is an arbitrary function which is continuous at $x = 0$,

then, $f(x) \delta x = 0$ if $x \neq 0$.

So, we can write $f(x) \delta(x) = f(0) \delta(x)$.

$$\text{As a result } \int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0) \int_{-\infty}^{\infty} \delta(x) dx = f(0)$$

Also if we shift from $x = 0$ to $x = a$ then dirac-delta function is represented by,

$$\delta(x-a) = 0 \text{ if } x \neq a \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x-a) dx = 1$$

$$= \infty \text{ if } x = a$$

In this case we have $\int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a)$.

Note : The range of integration need not be always from $-\infty$ to ∞ but may take any value about the singular point at which the delta function is not zero.

2.2 REPRESENTATION OF DIRAC-DELTA FUNCTION

(a) As limit of Gaussian Function.

The Gaussian function is represented as $G_{\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-x')^2}{2\sigma^2}}$.

such that, $\int_{-\infty}^{+\infty} G_{\sigma} dx = 1$ (unit area under the curve).

Then, $\delta(x-x') =$

$$\lim_{\sigma \rightarrow 0} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-x')^2}{2\sigma^2}}$$

We have,

$$\int_{-\infty}^{+\infty} G_{\sigma} dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(x-x')^2}{2\sigma^2}} dx$$

$$\left(\text{Let us put } y = \frac{x-x'}{\sqrt{2}\sigma} \right)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-y^2} \sigma\sqrt{2} dy$$

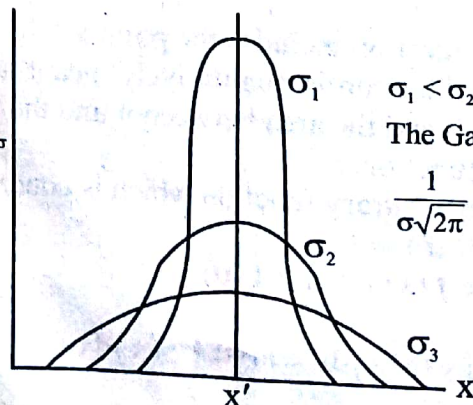
$$(dx = \sigma\sqrt{2} dy)$$

$$= \frac{\sigma\sqrt{2}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-y^2} dy = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1 \quad \left(\because \int_{-\infty}^{+\infty} e^{-y^2} dy = \sqrt{\pi} \right)$$

Hence in the limit $\sigma \rightarrow 0$ $G_{\sigma}(x)$ has properties of delta function that is

$$\delta(x-x') = \lim_{\sigma \rightarrow 0} \frac{1}{\sigma\sqrt{2\pi}} \left[e^{-\frac{(x-x')^2}{2\sigma^2}} \right] \sigma > 0$$

$$\text{If } x' = 0 \text{ then } \delta(x) = \lim_{\sigma \rightarrow 0} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$



$$\sigma_1 < \sigma_2 < \sigma_3$$

The Gaussian function has value

$$\frac{1}{\sigma\sqrt{2\pi}} \text{ at } x = x' \text{ and width } 2\sigma.$$

(The delta function as limit of Gaussian function)

(b) As limit of Rectangular Function.

The Rectangular function is given by the equation,

$$R_\sigma = \frac{1}{2\sigma} \text{ for } -\sigma < x - x' < \sigma$$

$$= 0 \text{ for } |x - x'| > \sigma.$$

So if we consider the limit $\sigma \rightarrow 0$ then the function $R_\sigma \rightarrow \infty$ and has value zero elsewhere.

$$\text{Now } \int_{-\infty}^{+\infty} R_\sigma dx = \int_{-\infty}^{+\infty} \frac{1}{2\sigma} dx = \frac{1}{2\sigma} \int_{x'-\sigma}^{x'+\sigma} dx$$

$$= \frac{1}{2\sigma} (x' + \sigma - x' + \sigma) = 1. \text{ [Independent of value of } \sigma]$$

The area under the curve remains unity. So the rectangular function R_σ has properties of dirac delta function.

$$\text{Then } \delta(x - x') = \lim_{\sigma \rightarrow 0} R_\sigma.$$

2.3 OTHER REPRESENTATION OF DIRAC-DELTA FUNCTION

(1) The Integral representation of delta function is given by $\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-x')} dx$.

$$(2) \delta(x - x') = \lim_{k \rightarrow \infty} \frac{\sin k(x - x')}{\pi(x - x')}$$

$$(3) \delta(x - x') = \lim_{k \rightarrow \infty} \frac{\sin^2 k(x - x')}{\pi k(x - x')^2}$$

$$(4) \delta(x) = \frac{1}{\pi} \lim_{n \rightarrow 0} \frac{n}{x^2 + n^2}$$

Properties of Dirac-Delta Function :

$$(1) \delta(x) = \delta(-x)$$

If $f(x)$ is any continuous and differentiable function then we have,

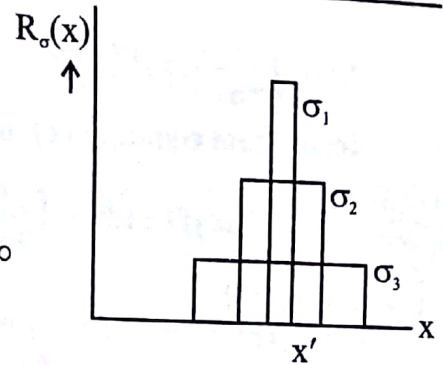
$$\int_{-\infty}^{+\infty} f(x) \delta(x) dx = f(0). \quad \dots\dots(1)$$

$$\text{Then, } \int_{-\infty}^{+\infty} f(x) \delta(-x) dx = \int_{-\infty}^{+\infty} f(-t) \delta(t) dt = f(0) \quad \dots\dots(2)$$

$\therefore$ From equation (1) and (2) it follows that $\delta(x) = \delta(-x)$.

$$(2) \delta(ax) = \frac{1}{|a|} \delta(-x)$$

$$\text{We consider } \int_{-\infty}^{+\infty} \delta(ax) f(x) dx = \frac{1}{a} \int_{-\infty}^{+\infty} \delta(t) f\left(\frac{t}{a}\right) dt \quad \left\{ \text{Putting } x = \frac{t}{a} \right\} = \frac{1}{a} f(0) \quad \dots\dots(1)$$



Delta function as limit of Rectangular function

$$\text{Also } \int_{-\infty}^{+\infty} \frac{1}{|a|} \delta(x) f(x) dx = \frac{1}{|a|} \int_{-\infty}^{+\infty} \delta(x) f(x) dx = \frac{1}{|a|} f(0) = \frac{1}{a} f(0) \quad \dots\dots(2)$$

Hence from equation (1) and (2) we get,

$$\int_{-\infty}^{+\infty} \delta(ax) f(x) dx = \int_{-\infty}^{+\infty} \frac{1}{|a|} \delta(x) f(x) dx \text{ then } \delta(ax) = \frac{1}{|a|} \delta(x)$$

$$(3) \quad x \delta(x) = 0$$

Let $f(x)$ be continuous. Let $x f(x) = F(x)$.

$$\text{Then } \int_{-\infty}^{+\infty} x \delta(x) f(x) dx = \int_{-\infty}^{+\infty} \delta(x) F(x) dx = F(0) = 0.$$

$$\Rightarrow x \delta(x) = 0$$

$$(4) \quad f(x) \delta(x-a) = f(a) \delta(x-a)$$

$$\text{We have } \int_{-\infty}^{+\infty} f(x) \delta(x-a) dx = f(a)$$

$$\text{Also } \int_{-\infty}^{+\infty} f(a) \delta(x-a) dx = f(a)$$

$$\text{We get, } f(x) \delta(x-a) = f(a) \delta(x-a).$$

$$(5) \quad \delta(x^2 - a^2) = \frac{1}{2a} [\delta(x-a) + \delta(x+a)] \quad a > 0$$

$$\text{We consider the integral } \int_{-\infty}^{+\infty} f(x) \delta(x^2 - a^2) dx$$

$$= \int_{-\infty}^0 f(x) \delta(x^2 - a^2) dx + \int_0^{+\infty} f(x) \delta(x^2 - a^2) dx \quad \dots\dots(1)$$

$$\text{Let us put } x^2 - a^2 = t \Rightarrow x^2 = a^2 + t \Rightarrow x = \pm \sqrt{t+a^2}$$

$$dx = \pm \frac{1}{2} (t+a^2)^{-\frac{1}{2}} (dt)$$

Then equation (1) takes the form as

$$\int_{-\infty}^{+\infty} f(x) \delta(x^2 - a^2) dx = \int_{-a^2}^{\infty} \frac{f(-\sqrt{t+a^2})}{2\sqrt{t+a^2}} \delta(t) dt + \int_{-a^2}^{\infty} \frac{f(\sqrt{t+a^2})}{2\sqrt{t+a^2}} \delta(t) dt$$

$$\text{i.e., } \int_{-\infty}^{+\infty} f(x) \delta(x^2 - a^2) dx = \frac{f(-a)}{2a} + \frac{f(a)}{2a} \quad \dots\dots(2)$$

$$\text{Also we have } \frac{1}{2a} \int_{-\infty}^{+\infty} f(x) [\delta(x-a) + \delta(x+a)] dx$$

$$= \frac{1}{2a} \left[\int_{-\infty}^{+\infty} f(x) \delta(x-a) dx + \int_{-\infty}^{+\infty} f(x) \delta(x+a) dx \right]$$

$$= \frac{1}{2a} [f(a) + f(-a)]$$

.....(3)

Comparing equation (2) and (3) we get,

$$\delta(x^2 - a^2) = \frac{1}{2a} [\delta(x - a) + \delta(x + a)]$$

$$(6) \int_{-\infty}^{+\infty} \delta(x - a) \delta(x - b) dx = \delta(a - b)$$

Let $f(a)$ be an arbitrary function, so that multiplying it on both sides of above equation and integrating with respect to a from $-\infty$ to ∞ we get,

$$\int_{-\infty}^{+\infty} da f(a) \int_{-\infty}^{+\infty} \delta(x - a) \delta(x - b) dx = \int_{-\infty}^{+\infty} da f(a) \delta(a - b) \quad \text{.....(1)}$$

Now consider the LHS of equation (1), i.e.,

$$\begin{aligned} \int_{-\infty}^{+\infty} da f(a) \int_{-\infty}^{+\infty} \delta(x - a) \delta(x - b) dx &= \int_{-\infty}^{+\infty} dx \delta(x - b) \int_{-\infty}^{+\infty} da f(a) \delta(x - a) \\ &= \int_{-\infty}^{+\infty} dx \delta(x - b) f(x) = f(b) \quad \text{.....(2)} \end{aligned}$$

$$\text{Now we can write RHS of equation (1) as } \int_{-\infty}^{+\infty} da f(a) \delta(a - b) = f(b) \quad \text{.....(3)}$$

Hence from equation (1), (2) and (3)

$$\text{We get, } \int_{-\infty}^{+\infty} \delta(x - a) \delta(x - b) dx = \delta(a - b).$$

Example 1 : Using δ Function Express mass or charge density function.

(a) Mass 4 at $x = -2$, and mass 5 at $x = -6$.

(b) Charge 2 at $x = -4$ and charge -4 at $x = 5$

Solution. A point mass or point charge can be assumed to be limiting case of function which is zero everywhere except at point $x = a$. So using δ function the solution of

(a) is $4\delta(x - 2) + 5\delta(x + 6)$ and that of (b) is $2\delta(x + 4) - 4\delta(x - 5)$.

Example 2 : Find the value of $\int_0^{\pi} \sin x \delta\left(x - \frac{\pi}{2}\right) dx$.

Solution. We know that $\int_a^b f(x) \delta(x - x') dx = f(x')$ for $a < x' < b$ otherwise

$$\therefore \int_0^{\pi} \sin x \delta\left(x - \frac{\pi}{2}\right) dx = \sin \frac{\pi}{2} = 1 \text{ as } 0 < \frac{\pi}{2} < \pi.$$

Example 3 : Evaluate $\int_{-1}^1 9x^3 \delta(3x + 1) dx$.

Solution.
$$\int_{-1}^1 qx^3 \delta(3x+1) dx = \int_{-1}^1 9x^3 \delta\left\{3\left(x + \frac{1}{3}\right)\right\} dx$$

$$= \int_{-1}^1 9x^3 \frac{1}{3} \delta\left(x + \frac{1}{3}\right) dx \quad \left\{ \because \delta(ax) = \frac{1}{a} \delta(x) \right\}$$

$$= 9 \left(-\frac{1}{3}\right)^3 \cdot \frac{1}{3} = -\frac{1}{9}.$$

Example 4 : Evaluate $\int_2^6 (2x^2 - 3x - 1) \delta(x-3) dx$.

Solution.
$$\int_2^6 (2x^2 - 3x - 1) \delta(x-3) dx = \int_2^6 2x^2 \delta(x-3) dx - \int_2^6 3x \delta(x-3) dx - \int_2^6 \delta(x-3) dx$$

$$= 2(3)^2 - (3)(3) - 1$$

$$= 28 - 9 - 1 = 8.$$

Example 5 : Evaluate $\int_0^3 t^3 \delta(t-5) dt$.

Solution. The Upper limit of Integration is 3 But the diradelta function at $t_0 = 5$ is outside the domain of integration.

So the value of $\int_0^3 t^3 \delta(t-5) dt = 0$. (By property of Dira Delta Function)

□

EXERCISE

Prove the following properties of Dirac-Delta function.

1. $\delta'(x) = -\delta'(-x)$

2. $x\delta'(x) = -\delta(x)$

3. $\int_{-\infty}^{+\infty} f(t)\delta'(t-a)dt = -f'(a)$

4. Prove that $\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-x')}dx$

5. Prove that $\delta(x) = \lim_{\alpha \rightarrow \infty} \frac{\sin \alpha x}{\pi x}$

6. Evaluate $\int_0^{\infty} e^{-3t}(\delta-4)dt$

7. Evaluate $\int_{-\infty}^{+\infty} \sin 2t \delta\left(t - \frac{\pi}{4}\right)dt$

8. $\int_{-\infty}^{\infty} f(x)\delta''(x-a)dx = f''(a)$

9. $\int_0^3 (5x-2)\delta(x-2)dx = 8$

10. $\int_{-2}^2 (2x+3)\delta(3x)dx = 1$

11. $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\vec{r})$

□

Example - 15 : If a force $\vec{F} = -3\hat{i} + 2\hat{j} + 2\hat{k}$ acts as a point $2\hat{i} + 3\hat{j} + 4\hat{k}$, find the moment of force about the origin.

Answer : Moment of force is $\left(\vec{r} \times \vec{F}\right) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times (-3\hat{i} + 2\hat{j} + 2\hat{k}) = -2\hat{i} - 16\hat{j} + 13\hat{k}$.

Example - 16 : A rigid body is rotating with angular velocity 5 radian per second about an axis parallel to $4\hat{i} - 2\hat{j} + \hat{k}$ through the point $2\hat{i} - \hat{j} + \hat{k}$. Find the magnitude of the velocity of point of rigid body at $3\hat{i} + 2\hat{j} - \hat{k}$.

Answer : The position vector is given by

$$\vec{r} = (3\hat{i} + 2\hat{j} - \hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = \hat{i} + 3\hat{j} - 2\hat{k}$$

$$\text{Angular velocity } \vec{\omega} = \omega \hat{n} = 5 \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{4^2 + (-2)^2 + (1)^2}}$$

$$\text{or } \vec{\omega} = \frac{5(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$$

So, velocity of the point $\vec{v} = \vec{\omega} \times \vec{r}$.

$$= \frac{5}{\sqrt{21}} (4\hat{i} - 2\hat{j} + \hat{k}) \times (\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{5}{\sqrt{21}} (\hat{i} + 9\hat{j} + 14\hat{k})$$

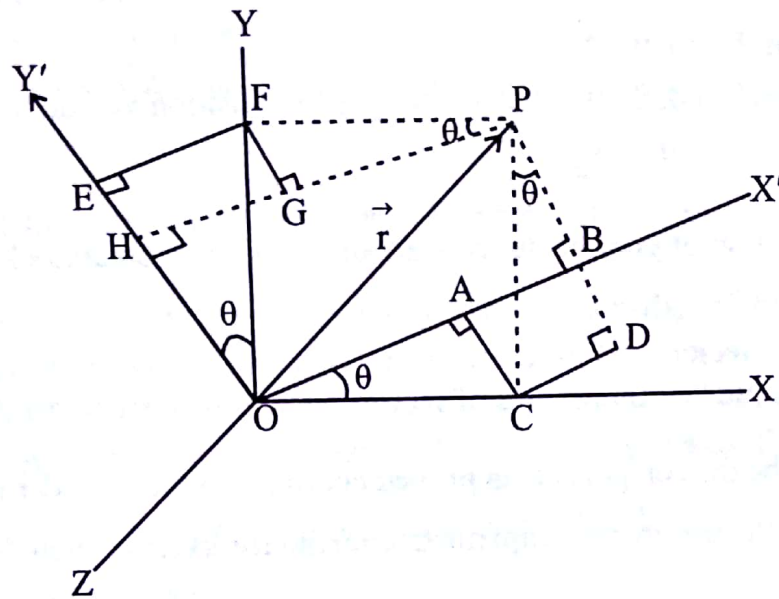
Its magnitude is given by, $V = \frac{5}{\sqrt{21}} \sqrt{1 + 81 + 196} = \frac{5}{\sqrt{21}} \sqrt{278} = 18.2$.

4.9 ROTATION OF CO-ORDINATE AXES : (PROPERTIES OF VECTOR UNDER ROTATION)

Let us discuss the manner in which a vector transforms under rotation. Let us consider the three dimensional cartesian coordinate system (x, y, z). Let the rotation be around z axis i.e x, y co-ordinates are rotated counter clockwise through angle θ keeping position vector $\vec{r}$ fixed. Let

$\vec{OP} = \vec{r}$ where P is the point with the coordinates x, y and z, such that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Similarly let x' , y' and z' be the co-ordinates of the point when axis is rotated by an angle θ in the new coordinate system (primed system). It is shown in the figure below.



In ΔOAC , $\cos \theta = \frac{OA}{OC}$ or $OA = OC \cos \theta$ $OA = x \cos \theta$

In ΔCDP , $\sin \theta = \frac{CD}{PC}$ or $CD = PC \sin \theta$ $CD = y \sin \theta$ But $AB = CD$.

Hence $OB = OA + AB$ i.e., $OB = x \cos \theta + y \sin \theta$.

or $x' = x \cos \theta + y \sin \theta$ (1)

as $OB = x'$ along X' axis.

Now in ΔOEF , $\cos \theta = \frac{OE}{OF}$ or $OE = OF \cos \theta$ or $OE = y \cos \theta$ as $OF = y$.

Also in ΔFPG , $\sin \theta = \frac{FG}{PF}$ or $FG = PF \sin \theta$

or $FG = x \sin \theta$. But $FG = EH$.

Then $OH = OE - EH$ or $y' = y \cos \theta - x \sin \theta$.

or $y' = -x \sin \theta + y \cos \theta$ (2) as $OH = y'$ along Y' axis.

Finally $z' = z$ (3)

Hence equation (1), (2) and (3) can be written as

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or $\vec{r}' = R(\theta) \vec{r}$

.....(4) using $\vec{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ and $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $R(\theta) = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which changes the position vector.

$\vec{r}$ to $\vec{r}'$ under rotation of coordinate axes about z-axis in counterclockwise sense.

So, $R(\theta)$ is called Transformation matrix or rotation matrix.

Also we know that vector can be represented by coordinates of a point which are proportional to the vector components. So the components of vector should transform under rotation of coordinate axes. So let (A_x', A_y') be the components in primed coordinate system under rotation and (A_x, A_y) be the components of vector in the unprimed coordinate system, then following equations (1) and (2) we have,

$$A_x' = A_x \cos \theta + A_y \sin \theta. \quad \dots\dots(5)$$

$$A_y' = -A_x \sin \theta + A_y \cos \theta. \quad \dots\dots(6)$$

Component of A along Z axis remains unchanged.

$$\text{i.e., } A_z' = A_z.$$

In terms of x and y components we find the magnitude of $\vec{A}$ as $A = \sqrt{A_x^2 + A_y^2} \quad \dots\dots(7)$

Also under rotation of coordinate system the new components of vector are A_x' and A_y' so that the magnitude of the new vector under rotation about z-axis will be given by $\sqrt{A_x'^2 + A_y'^2}$.

$$= \sqrt{(A_x \cos \theta + A_y \sin \theta)^2 + (-A_x \sin \theta + A_y \cos \theta)^2}$$

$$= \sqrt{A_x^2 + A_y^2} \text{ which is same as in equation (7).}$$

Hence though the components of a vector may change under rotation by some angle, but the magnitude of the vector in the unprimed and primed coordinate system remains invariant. (The magnitude of the vector is a scalar quantity and so it must be invariant under rotation of coordinate axes. This is an important property of vector. Also it can be shown that the direction of the given vector $\vec{A}$ remains unchanged or invariant under rotation of the coordinate axes though the components of vector may change to new values. Hence the vector $\vec{A}$ is independent of the rotation of the coordinate system. The vector equations are independent of any particular coordinate system. (The coordinate system may be cartesian or any other system).

Invariance of Scalar product of Two Vector under Rotation :

Scalars are independent of the coordinate system. Let us examine the behaviour of scalar

product of Two Vectors $\vec{A} \cdot \vec{B}$ under rotation of the coordinate axes.

If we consider the rotation of the cartesian coordinate system about Z-axis by angle θ then as discussed in previous article, the transformation of the components of vector $\vec{A}$ under rotation of x y axes through angle θ about Z-axis are given as

$$\begin{aligned}A'_x &= A_x \cos \theta + A_y \sin \theta. \\A'_y &= -A_x \sin \theta + A_y \cos \theta. \\A'_z &= A_z.\end{aligned}$$

Similarly the components of vector $\vec{B}$ under rotation are represented as

$$\begin{aligned}B'_x &= B_x \cos \theta + B_y \sin \theta. \\B'_y &= -B_x \sin \theta + B_y \cos \theta. \\B'_z &= B_z.\end{aligned}$$

Hence as A_x , A_y and A_z are components of $\vec{A}$ along x, y, z axes then

$\vec{A} \cdot \vec{B} = (A_x B_x + A_y B_y + A_z B_z)$ as obtained earlier in the discussion of scalar product of Two Vectors in terms of its components.

So, let us find scalar product of $\vec{A}$ and $\vec{B}$ in terms of new components (A'_x, A'_y, A'_z) and (B'_x, B'_y, B'_z) under rotation.

The result is $A'_x B'_x + A'_y B'_y + A'_z B'_z$.

$$\begin{aligned}&= (A_x \cos \theta + A_y \sin \theta) (B_x \cos \theta + B_y \sin \theta) \\&\quad + (-A_x \sin \theta + A_y \cos \theta) (-B_x \sin \theta + B_y \cos \theta) + A_z B_z. \\ \text{or } A'_x B'_x + A'_y B'_y + A'_z B'_z &= A_x B_x \cos^2 \theta + A_x B_y \cos \theta \sin \theta + A_y B_x \sin \theta \cos \theta + A_y B_y \sin^2 \theta \\&\quad + A_x B_x \sin^2 \theta - A_x B_y \sin \theta \cos \theta - A_y B_x \cos \theta \sin \theta + A_y B_y \cos^2 \theta + A_z B_z \\&= A_x B_x (\cos^2 \theta + \sin^2 \theta) + A_y B_y (\cos^2 \theta + \sin^2 \theta) + A_z B_z\end{aligned}$$

$$\text{or } \boxed{A'_x B'_x + A'_y B'_y + A'_z B'_z = A_x B_x + A_y B_y + A_z B_z}$$

So, scalar product of Two Vectors is invariant under rotation of coordinate axes.

4.10 SCALAR TRIPLE PRODUCT :

The scalar triple product of three vectors is the scalar product of a vector with vector product of the other two vectors. If, $\vec{A}, \vec{B}$ and $\vec{C}$ be any three vectors then examples of scalar triple

product are $\vec{A} \cdot (\vec{B} \times \vec{C}), (\vec{A} \times \vec{B}) \cdot \vec{C}, (\vec{B} \times \vec{C}) \cdot \vec{A}, \vec{C} \cdot (\vec{A} \times \vec{B})$ etc.

It is also represented as $\left[\vec{A} \vec{B} \vec{C} \right]$ which is a scalar quantity.

Geometrical Interpretation :

The scalar triple product $\vec{A} \cdot (\vec{B} \times \vec{C})$ represents the volume of a parallelopiped having

$\vec{A}, \vec{B}$ and $\vec{C}$ as its coterminus edges.

Figure below shows a parallelopiped with $\vec{OP} = \vec{A}, \vec{OQ} = \vec{B}, \vec{OS} = \vec{C}$, as coterminus edges.

The altitude of the parallelopiped is $h = A \cos \phi$.

The volume of the parallelopiped.

= (Area of its base) (vertical height or altitude).

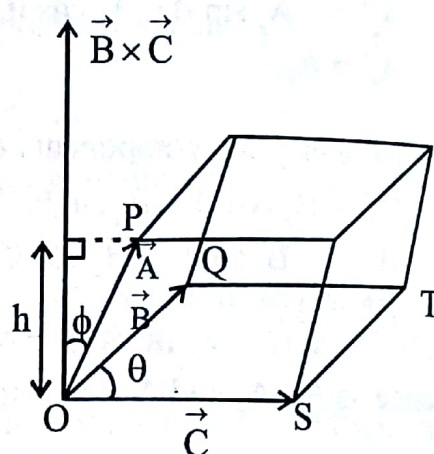
= (Area of parallelogram OSTQ) h.

$$= |\vec{B} \times \vec{C}| (A \cos \phi)$$

$$= AD \cos \phi \text{ where } \vec{D} = (\vec{B} \times \vec{C}) \text{ (let)}$$

$$= \vec{A} \cdot \vec{D} \quad \therefore D = |\vec{B} \times \vec{C}|$$

$$= \vec{A} \cdot (\vec{B} \times \vec{C})$$



This value of scalar triple product is positive if ϕ is an acute angle. (If $\vec{A}, \vec{B}, \vec{C}$ form a right handed system of vectors.)

Also this value is negative if ϕ is obtuse i.e., $\vec{A}, \vec{B}, \vec{C}$ form a left handed system.

Characteristics of Scalar Triple Product :

1. Condition for three vectors to be coplanar :

If $\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar, then value of scalar triple product is zero. In this case volume of the parallelopiped becomes zero. i.e., $[\vec{A} \vec{B} \vec{C}] = 0$.

2. If any Two vectors of a scalar triple product are equal then the product is zero.

$$\text{i.e., } [\vec{A}, \vec{A}, \vec{B}] = [\vec{A}, \vec{B}, \vec{B}] = [\vec{A}, \vec{C}, \vec{C}] \text{ etc. vanish.}$$

$$\text{As } [\vec{A}, \vec{B}, \vec{B}] = \vec{A} \cdot (\vec{B} \times \vec{B}) = \text{zero.}$$

3. If two of the vectors are parallel the value of scalar triple product is zero. So if $\vec{B}$ is parallel to $\vec{A}$ i.e., $\vec{B} = K\vec{A}$ where K is a scalar then $[\vec{A} \vec{B} \vec{C}] = (\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{A} \times K\vec{A}) \cdot \vec{C} = 0$ as $\vec{A} \times \vec{A} = 0$.

4. The value of scalar triple product depends on the cyclic order of vectors.

Any one face of the parallelopiped may be taken as its base so volume of parallelopiped may also be given as $\vec{B} \cdot (\vec{C} \times \vec{A})$ and $\vec{C} \cdot (\vec{A} \times \vec{B})$ i.e., cyclic order of $\vec{A}, \vec{B}$ and $\vec{C}$ are maintained.

$$\text{i.e., } [\vec{A} \vec{B} \vec{C}] = [\vec{B} \vec{C} \vec{A}] = [\vec{C} \vec{A} \vec{B}] = V \text{ the volume of parallelopiped.}$$

$$\text{But } \vec{A} \cdot (\vec{C} \times \vec{B}) = \vec{A} \cdot \left\{ -(\vec{B} \times \vec{C}) \right\} = -\vec{A} \cdot (\vec{B} \times \vec{C})$$

$$\text{Hence } [\vec{A} \vec{C} \vec{B}] = [\vec{B} \vec{A} \vec{C}] = [\vec{C} \vec{B} \vec{A}] = -V.$$

5. The value of scalar triple product is independent of the position of dot and cross. In other words the position of dot and cross may be interchanged without changing the value if the cyclic order of vectors is maintained.

$$\text{We have } \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\text{i.e., } \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\text{or } \vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$$

Hence the position of dot and cross can be interchanged in a scalar triple product.

6. Scalar triple product in terms of components.

$$\text{Let } \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}, \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}, \vec{C} = C_x \hat{i} + C_y \hat{j} + C_z \hat{k}$$

$$\text{Then } (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= (B_y C_z - B_z C_y) \hat{i} + (B_z C_x - B_x C_z) \hat{j} + (B_x C_y - B_y C_x) \hat{k}$$

$$\text{Then } \vec{A} \cdot (\vec{B} \times \vec{C}) = (\hat{i} A_x + \hat{j} A_y + \hat{k} A_z) \cdot [(B_y C_z - B_z C_y) \hat{i} + (B_z C_x - B_x C_z) \hat{j} + (B_x C_y - B_y C_x) \hat{k}]$$

$$\text{or } \vec{A} \cdot (\vec{B} \times \vec{C}) = A_x (B_y C_z - B_z C_y) + A_y (B_z C_x - B_x C_z) + A_z (B_x C_y - B_y C_x)$$

$$\text{or } \vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

Note : If $\vec{A}, \vec{B}$ and $\vec{C}$ lie in one plane i.e., they are coplanar then we know $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$.

$$\text{or } \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = 0 \text{ which is condition for the three vectors } \vec{A}, \vec{B} \text{ and } \vec{C} \text{ to be coplanar.}$$

7. Distributive law holds for scalar triple product,

$$\text{i.e., } [\vec{A}, \vec{B} + \vec{D}, \vec{C} + \vec{E}] = [\vec{A}\vec{B}\vec{C}] + [\vec{A}\vec{B}\vec{E}] + [\vec{A}\vec{D}\vec{C}] + [\vec{A}\vec{D}\vec{E}] \text{ because both the scalar}$$

and vector product of two vectors are distributive.

4.11 VECTOR TRIPLE PRODUCT

The vector triple product of three vectors is vector product of any vector with vector product of remaining two vectors. Examples of vector triple product are

$\vec{A} \times (\vec{B} \times \vec{C}), \vec{B} \times (\vec{C} \times \vec{A}), (\vec{A} \times \vec{B}) \times \vec{C}$ etc. The product is a vector quantity.

Expansion Formula : $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

Proof : $\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z, \vec{B} = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z, \vec{C} = \hat{i}C_x + \hat{j}C_y + \hat{k}C_z$

$$\vec{B} \times \vec{C} = (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) \times (\hat{i}C_x + \hat{j}C_y + \hat{k}C_z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \hat{i}(B_yC_z - B_zC_y) - \hat{j}(B_xC_z - B_zC_x) + \hat{k}(B_xC_y - B_yC_x)$$

$$\therefore \vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ (B_yC_z - B_zC_y) & (B_zC_x - B_xC_z) & (B_xC_y - B_yC_x) \end{vmatrix}$$

$$\Rightarrow \vec{A} \times (\vec{B} \times \vec{C}) = \hat{i}[A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)] + \hat{j}[A_z(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_x)] + \hat{k}[A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)] \quad \dots\dots(1)$$

Now we evaluate $(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$.

$$= (A_xC_x + A_yC_y + A_zC_z)(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) - (A_xB_x + A_yB_y + A_zB_z)(\hat{i}C_x + \hat{j}C_y + \hat{k}C_z)$$

$$= A_xB_xC_x\hat{i} + A_xB_yC_x\hat{j} + A_xB_zC_x\hat{k} + A_yB_xC_y\hat{i} + A_yB_yC_y\hat{j} + A_yB_zC_y\hat{k} + A_zB_xC_z\hat{i}$$

$$+ A_zB_yC_z\hat{j} + A_zB_zC_z\hat{k} - A_xB_xC_x\hat{i} - A_xB_xC_y\hat{j} - A_xB_xC_z\hat{k} - A_yB_yC_x\hat{i} - A_yB_yC_y\hat{j}$$

$$- A_yB_yC_z\hat{k} + A_zB_zC_x\hat{i} - A_zB_zC_y\hat{j} - A_zB_zC_z\hat{k}$$

$$\begin{aligned}
\text{i.e., } & \left(\vec{A} \cdot \vec{C} \right) \vec{B} - \left(\vec{A} \cdot \vec{B} \right) \vec{C} \\
&= \hat{i} A_y B_x C_y - \hat{i} A_y B_y C_x + \hat{i} A_y B_x C_y - \hat{i} A_z B_z C_x + \hat{j} A_z B_y C_z - \hat{j} A_z B_z C_y \\
&\quad + \hat{j} A_x B_y C_x - \hat{j} A_x B_y C_y + \hat{k} A_x B_z C_x - \hat{k} A_x B_x C_z + \hat{k} A_y B_z C_y - \hat{k} A_y B_y C_z \\
\text{i.e., } & \left(\vec{A} \cdot \vec{C} \right) \vec{B} - \left(\vec{A} \cdot \vec{B} \right) \vec{C} = \hat{i} [A_y (B_x C_y - B_y C_x) - A_z (B_z C_x - B_x C_z)] \\
&\quad + \hat{j} [A_z (B_y C_z - B_z C_y) - A_x (B_x C_y - B_y C_x)] + \hat{k} [A_x (B_z C_x - B_x C_z) - A_y (B_y C_z - B_z C_y)] \\
&\quad \dots\dots\dots (2)
\end{aligned}$$

From equation (1) and (2) we get,

$$\vec{A} \times (\vec{B} \times \vec{C}) = \left(\vec{A} \cdot \vec{C} \right) \vec{B} - \left(\vec{A} \cdot \vec{B} \right) \vec{C}. \quad (\text{Proved})$$

As an aid for memory we write

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\text{Dot product of first and third vector}) \text{ second vector} - (\text{Dot product of first}$$

and second vector) third vector.

Characteristics of Vector Triple Product :

1. It is not commutative i.e., $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{B} \times \vec{C}) \times \vec{A}$ as $\vec{A} \times (\vec{B} \times \vec{C}) = -(\vec{B} \times \vec{C}) \times \vec{A}$
(From property of cross product or vector product of two vectors).

2. It is not associative i.e., $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$.

$$\text{As we know } \vec{A} \times (\vec{B} \times \vec{C}) = \left(\vec{A} \cdot \vec{C} \right) \vec{B} - \left(\vec{A} \cdot \vec{B} \right) \vec{C}$$

$$\begin{aligned}
\text{But } (\vec{A} \times \vec{B}) \times \vec{C} &= -\vec{C} \times (\vec{A} \times \vec{B}) = -\left[\left(\vec{B} \cdot \vec{C} \right) \vec{A} - \left(\vec{A} \cdot \vec{C} \right) \vec{B} \right] \\
&= \left(\vec{A} \cdot \vec{C} \right) \vec{B} - \left(\vec{B} \cdot \vec{C} \right) \vec{A}.
\end{aligned}$$

3. Let $\vec{D} = \vec{B} \times \vec{C}$ so $\vec{D}$ is perpendicular to the plane containing $\vec{B}$ and $\vec{C}$. Now $\vec{A} \times (\vec{B} \times \vec{C})$ is a vector perpendicular to plane formed by $\vec{A}$ and $\vec{D}$ ($\vec{B} \times \vec{C}$). So $\vec{A} \times (\vec{B} \times \vec{C})$ will be in the plane formed by $\vec{B}$ and $\vec{C}$ and in this plane it is perpendicular to $\vec{A}$.